Appendix

Here we present an example, which applies our theory to improve a key theorem of O'Kane and Shell [1].

Theorem 1. *Let Λ_0 be a set of size at least* 2*. Let (S, Λ_0, T) be a finite transition system, and $\sigma\colon S \to L$ a labeling such that the range of σ has cardinality $N \in \mathbb{N}$, $N \geq 2$ and let $K \in \mathbb{N}$ be a number $K \geq \max\{N+2, 6\}$. The decision problem of whether there exists a sufficient refinement σ' of σ with at most K labels is NP-complete.*

Proof. We first prove this for $N = 6$ and $K = 8$. Note that if this is proved for some N_0, K_0, then it immediately follows for all $N > N_0$ and $K \geq N + K_0 - N_0$, because we can add extra elements to the transition system on the side whith a certain number of labels. We will give an argument for $2 \leqslant N < 6$ in the end of the proof.

The problem is in NP, because given a labeling σ' the requirements $\sigma' \succeq \sigma$, $|\operatorname{ran}\sigma'| \leqslant K$, and σ' is sufficient can be verified in (at most) quadratic time $O((|S| + |\Lambda_0| + |T|)^2)$. O'Kane and Shell [1] showed that this problem is NP-hard, but for them the set Λ of transition-labels was variable whereas for us it is fixed. To be more precise, let $n \in \mathbb{N}$. O'Kane and Shell showed that given a graph G of size n, there is a transition system $(S(G), \Lambda(G), T(G))$ which can be constructed in polynomial time from G, such that

$$|S(G) + \Lambda(G) + T(G)| \leqslant (3+n) + n^2 + 2n,$$

and a labeling $\sigma(G)\colon S(G) \to \{1,2,3,4\}$ such that G is three-colorable if and only if $\sigma(G)$ admits a sufficient refinement with at most 6 labels. Thus, they polynomially reduced the three-colorability problem on finite graphs to the problem of finding a sufficient refinement of size 6 on transition systems with an unbounded set of transition labels. We will show that the latter problem is polynomially reducible to the problem of finding a sufficient refinement of size 8 on transition systems with fixed Λ_0 of size at least 2. Without loss of generality, we may assume that Λ_0 actually has size exactly 2 and $\Lambda_0 = \{0, 1\}$.

So suppose that (S_1, Λ_1, T_1) is a transition system such that $(|S_1| + |\Lambda_1| + |T_1|) \leqslant n$, and $\sigma_1\colon S_1 \to \{1,2,3,4\}$ a labeling function. We will construct a transition system (S, Λ_0, T) such that

$$|S| \leqslant |S_1| + |T_1|m \text{ and } |T| \leqslant 2|T_1|m$$

in which

$$m = \lceil \log_2 |\Lambda_1| \rceil,$$

and a labeling function $\sigma\colon S \to \{-1, 0, 1, 2, 3, 4\}$ such that σ_1 has a refinement with 6 labels if and only if and σ has a refinement with 8 labels. From the above it follows that $|S| + |T| \leqslant n + 6n\log n \leqslant 7n\log n$, so in terms of size the reduction is polynomial. Let us now show the construction itself.

By the choice of m, there is a one-to-one map ρ from Λ_1 to the set Λ_0^m of all functions from $\{0, \dots, m-1\}$ to $\Lambda_0 = \{0, 1\}$. Let us construct S, T, and σ as follows:

1. for each $s_1 \in S_1$, add an element $\gamma(s_1)$ to S and define $\sigma(\gamma(s_1)) = \sigma_1(s_1)$.
2. add an extra element $s_* \in S$ and define $\sigma(s_*) = -1$,
3. for each $(s_1, \lambda, s_1') \in T_1$, do the following:
   - Add new elements $z_0, \dots, z_{m-2}$ to S,
   - Define $\sigma(z_i) = 0$ for all $i < m$,
   - Add the triples $(\gamma(s_1), \rho(0), z_0)$ and $(z_{m-2}, \rho(m-1), \gamma(s_1'))$ to T,
   - For each $i < m-2$ add the triple $(z_i, \rho(i+1), z_{i+1})$ to T.
   - For each $i < m-1$ add the triple $(s_*, 0, z_i)$ to T.

The idea is that the transition

$$s_1 \xrightarrow{\lambda} s_1'$$

is replaced by the sequence of transitions

$$\gamma(s_1) \overset{\rho(\lambda)(0)}{\longrightarrow} z_0 \overset{\rho(\lambda)(1)}{\longrightarrow}, \dots, \overset{\rho(\lambda)(m-2)}{\longrightarrow}, z_{m-2} \overset{\rho(\lambda)(m-1)}{\longrightarrow} \gamma(s_1'). \tag{5}$$

Thus, we used $m-1$ new elements for each transition in T_1 which is why $|S| < |S_1| + |T_1|m + 1$ and therefore $|S| \leqslant |S_1| + |T_1|m$, and $2m$ transitions for each transition in T_1 which is why $|T| \leqslant 2|T_1|m$.

We will now show that σ has a refinement with 8 classes iff σ_1 has a refinement with 6 classes. Suppose σ_1 has a sufficient refinement σ_1' with 6 classes. Without loss of generality we may assume that $\mathrm{ran}(\sigma_1') = \{1, 2, 3, 4, 5, 6\}$. Then define $\sigma'(\gamma(s)) = \sigma_1'(s)$ for all $s \in S_1$ and if $z \in S \setminus \{\gamma(s_1) \mid s_1 \in S_1\}$, then define $\sigma'(z) = \sigma(z)$. By the way transitions were replaced (5), one can see that σ' is a sufficient refinement of σ with labels $\{-1, 0, 1, 2, 3, 4, 5, 6\}$ of which there are 8. Suppose σ' is a sufficient refinement of σ with 8 classes. Because of the way σ' was defined, all elements z_i added in clause (3) must have the same label (because otherwise transitions initiating at s_* would lead to a contradiction with sufficiency). Thus, we may assume without loss of generality that $\mathrm{ran}\,\sigma' = \{-1, 0, 1, 2, 3, 4, 5, 6\}$ and the labels of z_i and of s_* are unchanged. Now define $\sigma_1'(s_1) = \sigma'(\gamma(s_1))$ for all $s_1 \in S_1$ in which case σ_1' must be a sufficient refinement of σ_1.

We must now give an argument why the Theorem is true also for $2 \leqslant N < 6$. In order to do that, we have to look more carefully at the original construction of O'Kane and Shell [1]. They used 4 base colors in their construction, but they could have used only 2. To be precise, the vertices v_0 and v_R can have the same color and the vertex v_L can have the same color as all the rest of the vertices v_i. Our construction adds two more colors -1 and 0, so we are down to $N \geq 4$, colors $\{-1, 0, 1, 2\}$. We can now replace -1 by 1 and 0 by 2 in the base coloring σ which will not influence its sufficient refinements, because in any sufficient refinement, the elements z_i must receive a separate color anyway, because otherwise there will be a contradiction, if in the original transition system there were any two transitions with the same transition label to two differently colored nodes. If there wasn't, add such in the beginning of the construction. Similarly s_* can be forced to receive a different color from any other nodes in any sufficient refinement. In this way for any $N < 4$ the problem will be equivalent to $N = 4$ which is why have the condition that K is at least $4 + 2 = 6$ in case N is too small. □

Remark The above theorem was proved for Λ_0 of size at least 2, but is true even for Λ_0 of size 1 (which is the best possible). To see this, use a similar trick as in the proof above to reduce transition systems with two transition labels $\Lambda = \{\lambda_0, \lambda_1\}$ to transition systems with only one label $\Lambda_* = \{\lambda_*\}$ by replacing each transition

$$s \xrightarrow{\lambda_i} s', \quad i \in \{0, 1\}$$

by a pair of transitions

$$s \xrightarrow{\lambda_*} z \xrightarrow{\lambda_*} s'$$

in which z is a new element whose color in the base labeling is i. Make sure that the labels of these new elements z cannot be refined by a similar trick as in the proof of the theorem above, that is, by adding an element s_* with outgoing connections to all such z.

References

1. J.M.O'Kane and D.A.Shell. Concise planning and filtering: Hardness and algorithms. *IEEE Transactions on Automation Science and Engineering, 14(4):1666-1681, 2017*